



# Percentile Methodology for Probability Distributions

## As Applied to the Representative Scenario Method

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## Introduction

Discussions on principle-based reserving approach for annuities include calculation of a modeled reserve. An alternative to a stochastic approach, the Representative Scenario Method (RSM) has been proposed. Instead of the thousands of scenarios used in the stochastic approach, the RSM creates a limited set of scenarios that are intended to be representative of the major risks affecting a given product group through Key Risk Drivers. For each Key Risk Driver, there are five scenarios with preset risk levels. This paper proposes a Percentiles Methodology for determining Key Risk Driver Reserves used in RSM.

For each Key Risk Driver, a set of five scenarios at different risk levels is determined. One of the five scenarios is the median, which is based on the company's anticipated experience. The other scenarios are based on regulatory preset risk levels, such as those mentioned in the 'Field Test Results' section of 'Annuity Reserve Field Test'<sup>1</sup>. Using scenario generator software, reserves are calculated for each scenario. Weights reflecting more probable scenarios are applied to the results to determine the Key Risk Driver Reserve.

The Percentiles Methodology uses the preset risk levels, experience of the company and the probability distribution of a Key Risk Driver to assign inputs for each scenario into the calculation engine. Additionally, the Percentiles Methodology uses a straight-forward method of weights to calculate the Key Risk Driver Reserve.

The appropriate distribution can vary for each Key Risk Driver. In this paper, the Normal Distribution, the Binomial Distribution, and the Poisson Distribution are used for renewal expenses, lapse, and mortality, respectively.

The next section of this paper will provide a technical description of the Percentile Methodology that can be applied to distributions and how to determine weights.

Subsequent sections of the paper will present examples of the Percentile Methodology for:

- Normal Distribution
- Binomial Distribution
- Poisson Distribution

[Appendix A](#) discusses problems presented by using a Standard Deviation Methodology in producing the desired risk level in percentiles and producing invalid results.

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<sup>1</sup> <https://www.soa.org/files/pd/2014/val-act/2014-valact-session-60.pdf>



[Appendix B](#) develops the formula for the Percentile Values of the Binomial Distribution which is used in step 3 of the Binomial Distribution Example of the Percentile Methodology.

[Appendix C](#) develops the Upper and Lower Bounds of the Poisson Distribution which are used in step 3 of the Poisson Distribution Example of the Percentile Methodology.

The inputs from this paper are designed to be used in determining the Scenario Reserves and Risk Driver Reserves of the Representative Scenarios Method. Steve Strommen has developed and explained a Reserve Generator that calculates those reserves. A publically available document is Christopher Olechowski's presentation on the 'Representative Scenario Methodology for Calculating Principle-Based Reserves'<sup>2</sup>.

[Appendix D: Academic Review](#) contains Jay Vadiveloo's review of this document.

## Technical Description of the Percentile Methodology

### **Step 1: Determine an appropriate probability density function (PDF) for that Key Risk Driver for the Key Risk Driver under consideration**

Estimate the parameter(s) (such as mean and standard deviation) of the probability distribution based on companies' historical data.

### **Step 2: Determine Scenario Risk Levels for Each Key Risk Driver**

The regulators could choose different scenario risk levels than those used in the Kansas Field Test and shown in the examples below. In the Kansas Field Test, 5 scenario risk levels were chosen (99.9%, 84%, 50%, 16%, and 0.1%). However, regulators could determine different scenario risk levels (e.g., choose 90% instead of the 99.9%).

### **Step 3: Calculate Percentile Values for Each Key Risk Driver's Scenario Risk Level**

Using the Key Risk Driver's probability distribution, one determines the percentile value for each of the 5 scenario risk levels chosen in Step 2. The formulas used to calculate the Percentile Values for each Scenario Risk Level varies based upon the probability distribution chosen.

Examples of determining the Percentile Values are shown in the Normal Distribution, Binomial Distribution and Poisson Distribution examples below.

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<sup>2</sup> <https://www.soa.org/Files/Pd/2014/annual-mtg/2014-orlando-annual-mtg-98-yt4.pdf>



#### **Step 4: Calculate Input for the Five Scenario Reserves for the Key Risk Driver under Consideration**

For each Key Risk Driver, there is a separate input to determine the Scenario Reserve for each of the 5 scenario risk levels chosen in step 2. Based on the Percentile Values calculated in Step 3, one calculates five separate inputs for the corresponding 5 scenarios. These inputs along with the median input values of all other Key Risk Drivers are used to calculate the reserves generated by the scenario generator that implements the Representative Scenario Method.

#### **Step 5: Determine the Boundaries of the Weight Calculation Range for the Five Scenario Reserves**

In order to develop weights for the five Scenario Reserves for each Key Risk Driver, one uses the Percentile Values set in Step 3. The boundaries of the Weight Calculation Range are the values that are halfway between the Percentile Values calculated in Step 3. If the distribution that was chosen in Step 1 is discrete and the results are not integers, round to the nearest integer.

#### **Step 6: Determine the cumulative distribution function (CDF) of the Boundaries Defining the Weight Calculation Range**

Determine the values of the CDF of the values determined in Step 5.

#### **Step 7: Calculate the Weight of Each Scenario**

Calculate the weight for each scenario by subtracting two consecutive values of the CDF determined in Step 6.

#### **Step 8: Calculate the Key Risk Driver Reserve**

For the Key Risk Driver under consideration, multiply each of the five Scenario Reserves generated using the inputs calculated in Step 4 by the corresponding weight from Step 7 and sum them up in order to determine the Key Risk Driver Reserve.



## Normal Distribution Example of the Percentile Methodology

The distribution of the average renewal expense per policy of one company is appropriate to be modeled as the Normal Distribution. If there is not sufficient support for an alternative distribution, the Normal Distribution is commonly used. The expectation is that an individual company's renewal expenses will follow a Normal Distribution.

### Methodology Steps:

**Step 1:** Determine the coefficients of the Normal Distribution based on the company's actual data.

Assume that, according to an experience study, the mean of the average renewal expense per policy ( $\mu$ ) is 100, and its standard deviation ( $\sigma$ ) is 10.

**Step 2:** Determine Scenario Risk Levels for Each Key Risk Driver.

Representative Scenario	Risk Levels
Scenario 1	99.9%
Scenario 2	84%
Scenario 3	50%
Scenario 4	16%
Scenario 5	0.1%

**Step 3:** Calculate the Percentile Value for each Key Risk Driver's Scenario Risk Level.

To determine Percentile Values for Expense, use the Normal Distribution. Calculate the Normal Percentile Values using the formulas shown below.

- Determine the Z Scores of the Standard Normal Distribution at each risk level.
- Calculate the Percentile Values of the Normal Distribution using the following formula.

$$\text{Percentile Value} = \mu + Z \times \sigma$$

Where,

$\mu$ : the mean of the average renewal expense per policy.

$\sigma$ : the standard deviation of the average renewal expense per policy.

Z: the Z score (can be positive or negative).

Risk Level	Z Score	Percentile Values
99.9%	3.0902	130.9023
84%	0.9945	109.9446*
50%	0	100
16%	-0.9945	90.0554
0.1%	-3.0902	69.0977

$$* 109.9446 = \mu + Z \times \sigma = 100 + 0.9945 \times 10 = 109.9446$$

(Rounded to 4 decimal places)

**Step 4:** Calculate Input for the five Scenario Reserves for the Key Risk Driver under consideration.

For this example, Percentile Values calculated in Step 3 can be used as the input for the Scenario Generator.

**Step 5:** Determine the Boundaries of the Weight Calculation Range for the Five Scenario Reserves.

The boundaries defining the Weight Calculation Range are the values that are halfway between the Percentile Values calculated in Step 3.

Risk Levels set in Step 2	Half-Way Value	Halfway between the Percentile Values at each risk level
84% and 99.9%	Half-Way Value 1 (H1)	120.4235*
50% and 84%	Half-Way Value 2 (H2)	104.9723
16% and 50%	Half-Way Value 3 (H3)	95.0277
0.1% and 16%	Half-Way Value 4 (H4)	79.5766

$$* 120.4235 = \frac{130.9023 + 109.9446}{2} \quad (\text{Rounded to 4 decimal places})$$

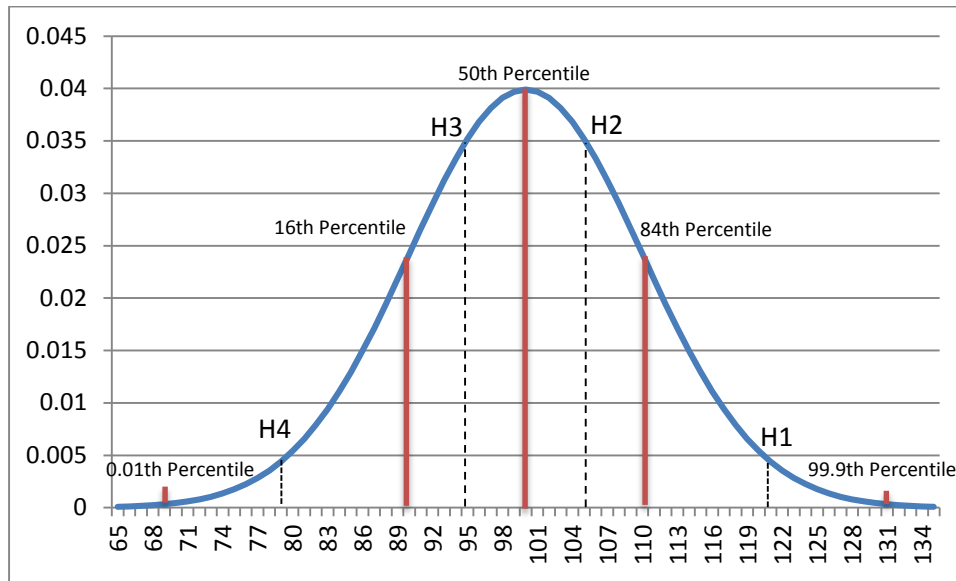
**Step 6:** Determine the Normal CDF of the boundaries defining the weight calculation range.

Risk Levels set in Step 2	Half-Way Value	Halfway between the Percentile Values at each risk level	Normal CDF
84% and 99.9%	H1	120	0.9772*
50% and 84%	H2	105	0.6915
16% and 50%	H3	95	0.3085
0.1% and 16%	H4	80	0.0228

$$* \Phi \left( \frac{\text{Halfway Value} - \text{Mean}}{\text{Standard Deviation}} \right)$$

$$= \Phi \left( \frac{120 - 100}{10} \right) = 0.9772$$

**Step 7:** Calculate the weight of each scenario.



Area > H1	Weight for 99.9% Scenario (1- CDF of H1)	0.0228
Area between H1 and H2	Weight for 84% Scenario (CDF of H1 – CDF of H2)	0.2858
Area between H2 and H3	Weight for 50% Scenario (CDF of H2 – CDF of H3)	0.3829
Area between H3 and H4	Weight for 16% Scenario (CDF of H3 – CDF of H4)	0.2858
Area ≤ H4	Weight for 0.1% Scenario (CDF of H4 – 0)	0.0228

**Step 8:** Calculate the Key Risk Driver Reserve.

The Expense Scenario Reserves at different risk levels are assumed as follows

Risk Level	Expense Scenario Reserve	Weight
99.9%	806,631,758	0.0228
84%	806,268,987	0.2858
50%	806,084,471	0.3829
16%	806,009,213	0.2858
0.1%	805,969,195	0.0228

$$\begin{aligned} \text{Expense Key Risk Driver Reserve} &= \sum (\text{Expense Scenario Reserve} \times \text{Unrounded Weight}) \\ &= 806,125,524 \end{aligned}$$





## Binomial Distribution Example of the Percentile Methodology

The probability distribution function of the number of lapses is assumed to be the Binomial Distribution. In considering lapse data, the results are either lapsed or not lapsed. This corresponds to a binomial situation where probabilities of a policy lapsing and not lapsing correspond to the binomial parameters  $p$  and  $q$ . For differing sizes of companies, the shape of the Binomial Distribution varies with the individual company's parameters.

### Methodology Steps:

**Step 1:** Determine the coefficients of the Binomial Distribution based on the company's actual data.

Assume that, according to an experience study, the total lapse policies exposed  $n = 10,000$ , and the number of lapses is 500, so the lapse rate  $p = 0.05$ .

**Step 2:** Determine Scenario Risk Levels for Each Key Risk Driver.

Representative Scenario	Risk Levels
Scenario 1	99.9%
Scenario 2	84%
Scenario 3	50%
Scenario 4	16%
Scenario 5	0.1%

**Step 3:** Calculate the Percentile Values for Each Key Risk Driver's Scenario Risk Level.

To determine Percentile Values for Lapse, use the Binomial Distribution. Calculate the Binomial Percentile Values using the formulas shown below.

- Determine the Z Scores of the Standard Normal Distribution at each risk level.
- Estimate the Percentile Values of the Binomial Distribution.

Use the following formula and rounding the results to the nearest integer:

$$\text{Percentile Value} = L + Z \times \sqrt{L \times (1 - \text{Lapse Rate})}$$

Where,

$L$ : Number of Lapses

$Z$ : The Z score (can be positive or negative)

Risk Level	Z Score	Percentile Values
99.9%	3.0902	567
84%	0.9945	522
50%	0	500
16%	-0.9945	478*
0.1%	-3.0902	433

$$* \text{Number of Lapses} - Z \text{ Score} \times \sqrt{\text{Number of Lapses} \times (1 - \text{Lapse Rate})}$$

$$478 \approx 500 - 0.9945 \times \sqrt{500 \times (1 - 0.05)} \quad (\text{Rounded to the nearest integer})$$

**Step 4:** Calculate Input for the Five Scenario Reserves for the Key Risk Driver under consideration

The inputs for the Scenario Generator to calculate the Lapse Key Risk Driver Reserve are done in three steps.

The first step is to determine the Percentile Values for Lapse at each risk level. For the Binomial Distribution this corresponds with the Percentile Values determined in Step 3. In the third step, this input varies by the 5 risk levels.

The second step is to use the company's actual policy exposure. In the third step, the company's actual policy exposure is a constant.

The third step is to calculate the input as:

$$Input_i = \frac{\text{Percentile Value}_i}{\text{company exposure}}$$

where,  $i$  varies from 1 to 5 (the number of risk levels)

For this example the company exposure is equal to 10,000.

Risk Level	Percentile Value	Input
99.9%	567	0.0567
84%	522	0.0522
50%	500	0.05
16%	478	0.0478*
0.1%	433	0.0433

$$* 0.0478 = \frac{478}{10,000}$$

**Step 5:** Determine the Boundaries of the Weight Calculation Range for the Five Scenario Reserves.

The boundaries defining the Weight Calculation Range are the values that are halfway between the Percentile Values calculated in Step 3.

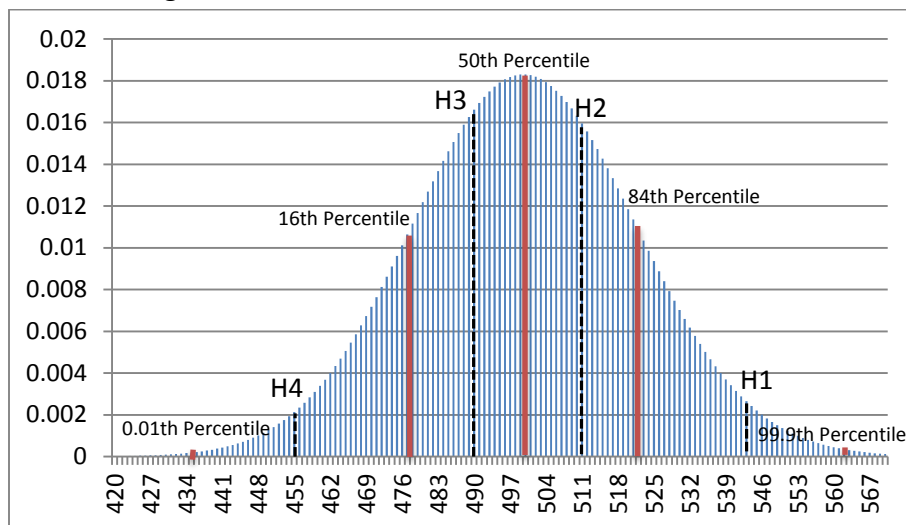
Risk Levels set in Step 2	Half-Way Value	Halfway between the Percentile Values at each risk level
84% and 99.9%	Half-Way Value 1 (H1)	545*
50% and 84%	Half-Way Value 2 (H2)	511
16% and 50%	Half-Way Value 3 (H3)	489
0.1% and 16%	Half-Way Value 4 (H4)	456

$$* 545 = \frac{567 + 522}{2} \text{ (Rounded to the nearest integer)}$$

**Step 6:** Determine the Binomial CDF of the boundaries defining the weight calculation range.

Risk Levels set in Step 2	Half-Way Value	Halfway between the Percentile Values at each risk level	Binomial CDF
84% and 99.9%	H1	545	0.9806
50% and 84%	H2	511	0.7028
16% and 50%	H3	489	0.3169
0.1% and 16%	H4	456	0.0218

**Step 7:** Calculate the weight of each scenario.



Area > H1	Weight for 99.9% Scenario ( 1- CDF of H1)	0.0194
Area between H1 and H2	Weight for 84% Scenario (CDF of H1 – CDF of H2)	0.2777
Area between H2 and H3	Weight for 50% Scenario (CDF of H2 – CDF of H3)	0.3860
Area between H3 and H4	Weight for 16% Scenario (CDF of H3 – CDF of H4)	0.2950
Area ≤ H4	Weight for 0.1% Scenario (CDF of H4 – 0)	0.0218

**Step 8:** Calculate the Key Risk Driver Reserve.

The Lapse Scenario Reserves for different risk levels are assumed as follows

Risk Level	Lapse Scenario Reserve	Weight
99.9%	810,379,648	0.0194
84%	807,784,580	0.2777
50%	806,084,471	0.3860
16%	804,670,130	0.2950
0.1%	800,589,276	0.0218

$$\begin{aligned} \text{Lapse Key Risk Driver Reserve} &= \sum (\text{Lapse Scenario Reserve} \times \text{Unrounded Weight}) \\ &= 806,102,861 \end{aligned}$$

## Poisson Distribution Example of the Percentile Methodology

The probability distribution function of the number of deaths is appropriate to be modeled as the Poisson Distribution. The Poisson Distribution assumes that the event that happens is rare, but its impact is significant which reflects both the frequency and the financial impact of a death claim. (In 1837, Siméon-Denis Poisson introduced the distribution that was later used to predict fatalities in the French army due to mule kicks<sup>3</sup>.) Since the Poisson Distribution deals with the occurrence of discrete events whose individual probability is low, one can derive more meaningful conclusions from low numbers of deaths within a block of business.

### Methodology Steps:

**Step 1:** Determine the coefficients of the Poisson Distribution based on the company's actual data.

Assume that, according to an experience study, the number of actual deaths for a company is 20. Therefore, its mortality can be modeled as the Poisson Distribution with Mean=Variance=20 ( $\lambda = 20$ ).

<sup>3</sup> The mule-kick example was examined in a book about the Poisson distribution titled 'The Law of Small Numbers'. The book was published by Ladislaus Bortkiewicz in 1898 ([http://en.wikipedia.org/wiki/Ladislaus\\_Bortkiewicz](http://en.wikipedia.org/wiki/Ladislaus_Bortkiewicz)).

**Step 2:** Determine Scenario Risk Levels for Each Key Risk Driver.

Representative Scenario	Risk Levels
Scenario 1	99.9%
Scenario 2	84%
Scenario 3	50%
Scenario 4	16%
Scenario 5	0.1%

**Step 3:** Calculate the Percentile Values for Each Key Risk Driver’s Scenario Risk Level.

To determine Percentile Values for Mortality, use the Poisson Distribution. Calculate the Poisson Percentile Values using the formula shown below.

- Determine the Z Scores of the Standard Normal Distribution at each risk level.
- Estimate the percentile values of the Poisson Distribution using the following formulas and rounding the results to the nearest integer.

$$Upper\ Bound = (A + 1) \times \left(1 - \frac{1}{9(A + 1)} + \frac{Z}{3\sqrt{A + 1}}\right)^3$$

$$Lower\ Bound = (A) \times \left(1 - \frac{1}{9A} + \frac{Z}{3\sqrt{A}}\right)^3$$

Where,

A: equals to the mean of the Poisson Distribution, Actual deaths.

Z: the Z score.

Upper Bound: the percentiles’ risk levels that are above or equal to 50%. For example, 60th percentile, 84th percentile.

Lower Bound: the percentiles’ risk levels that are below 50%. For example, 40th percentile, 16th percentile.

Risk Level	Z Score	Percentile Values
99.9%	3.0902	38
84%	0.9945	26*
50%	0	20
16%	-0.9945	16**
0.1%	-3.0902	9

$$* 26 = (20 + 1) \times \left(1 - \frac{1}{9(20 + 1)} + \frac{0.9945}{3\sqrt{20 + 1}}\right)^3$$

$$** 16 = 20 \times \left(1 - \frac{1}{9 \times 20} + \frac{-0.9945}{3\sqrt{20}}\right)^3$$

Percentile Values at the Scenario Risk Level are determined and rounded to the nearest integer.



**Step 4:** Calculate Input for the Five Scenario Reserves for the Key Risk Driver under consideration

The input for the Scenario Generator to calculate the Mortality Key Risk Driver Reserve is done in three steps.

The first step is to determine the projected number of deaths at each risk level mortality rate. For the Poisson Distribution this corresponds with the Percentile Values determined in Step 3. In the third step, this input varies by the 5 risk levels.

The second step is to determine the company expected deaths based upon a mortality rate from standard mortality table times the company’s actual policy exposure. In the third step, the company expected deaths is a constant

The third step is to calculate the input as:

$$Input_i = \frac{Percentile\ Value_i}{company\ expected\ deaths}$$

where, *i* varies from 1 to 5 (the number of risk levels).

The company expected deaths is calculated by summing the product of the mortality rate from a standard mortality table times the company’s exposure. This is a constant in the formula given above.

For each Risk Level, the Input is calculated by dividing the Percentile Value by company expected deaths.

Assume that for this example the number of company expected deaths is 18 (this number is the product of the company exposure times an industry table). Then the Inputs are:

Risk Level	Percentile Value	Input
99.9%	38	2.111
84%	26	1.444*
50%	20	1.111
16%	16	0.889
0.1%	9	0.500

$$* 1.444 \approx \frac{26}{18}$$

**Step 5:** Determine the Boundaries of the Weight Calculation Range for the Five Scenario Reserves.

The boundaries defining the Weight Calculation Range are the values that are halfway between the Percentile Values calculated in Step 3.

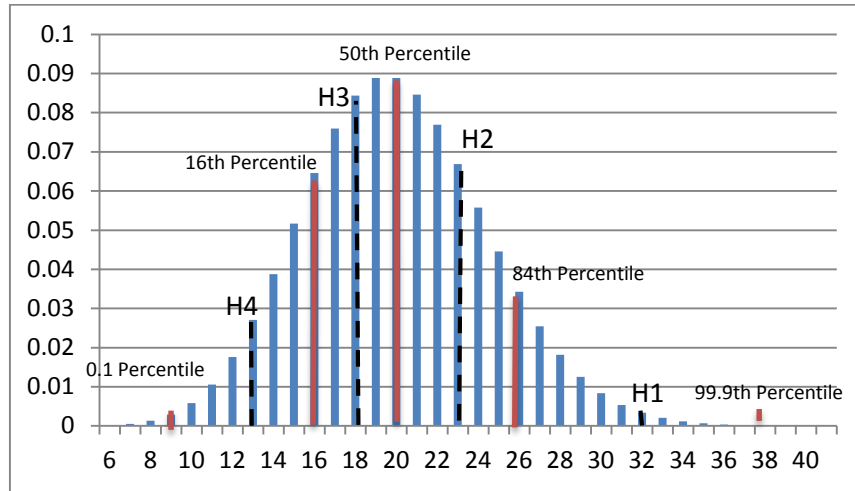
Risk Levels set in Step 2	Half-Way Value	Halfway between the Percentile Values at each risk level
84% and 99.9%	Half-Way Value 1 (H1)	32*
50% and 84%	Half-Way Value 2 (H2)	23
16% and 50%	Half-Way Value 3 (H3)	18
0.1% and 16%	Half-Way Value 4 (H4)	13

$$* 32 = \frac{38 + 26}{2} \quad (\text{Rounded to the nearest integer})$$

**Step 6:** Determine the Poisson CDF of the boundaries defining the weight calculation range.

Risk Levels set in Step 2	Half-Way Value	Halfway between the Percentile Values at each risk level	Poisson CDF
84% and 99.9%	H1	32	0.9953
50% and 84%	H2	23	0.7875
16% and 50%	H3	18	0.3814
0.1% and 16%	H4	13	0.0661

**Step 7:** Calculate the weight of each scenario.



Area > H1	Weight for 99.9% Scenario ( 1- CDF of H1)	0.0047
Area between H1 and H2	Weight for 84% Scenario (CDF of H1 – CDF of H2)	0.2078
Area between H2 and H3	Weight for 50% Scenario (CDF of H2 – CDF of H3)	0.4061
Area between H3 and H4	Weight for 16% Scenario (CDF of H3 – CDF of H4)	0.3153
Area ≤ H4	Weight for 0.1% Scenario (CDF of H4 – 0)	0.0661

**Step 8:** Calculate the Key Risk Driver Reserve.

The Mortality Scenario Reserves for different risk levels are assumed as follows

Risk Level	Mortality Scenario Reserve	Weight
99.9%	800,183,216	0.0047
84%	804,128,122	0.2078
50%	806,084,471	0.4061
16%	808,583,619	0.3153
0.1%	812,611,846	0.0661

$$\text{Mortality Key Risk Driver Reserve} = \sum (\text{Mortality Scenario Reserve} \times \text{Unrounded Weight}) = 806,869,691$$



## Appendix A: Standard Deviation Methodology: Problems with Risk Levels and Invalid Results

As shown in the examples below, using standard deviations may result in:

1. Standard deviations not producing the desired risk level in percentiles.
2. Adding or subtracting a number of Standard Deviations from the Mean can give invalid results.

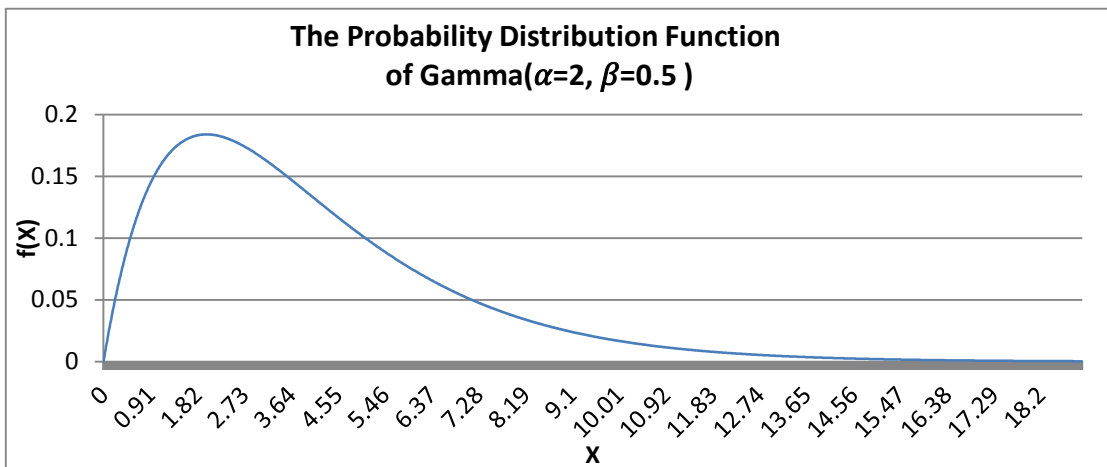
The following risk levels in percentiles were associated with standard deviations. This is shown in the following table:

	Risk Levels in Percentiles	Standard Deviations of Normal Distribution at Risk Levels
Scenario 1	99.9%	3 Standard Deviations
Scenario 2	84%	1 Standard Deviation
Scenario 3	50%	Mean
Scenario 4	16%	-1 Standard Deviation
Scenario 5	0.1%	-3 Standard Deviations

For Normal Distribution, the risk levels in percentile values are consistent with the values located in the standard deviation values at the same row listed above. However, for other distributions, especially for those that cannot be approximated by Normal Distribution, the risk levels in percentiles may correspond to different standard deviations.

### 1. For some distributions, the desired risk levels may not be produced by standard deviation.

For distributions other than the Standard Normal Distribution, using standard deviations may not produce the risk level in percentiles. This occurs because other distributions may not be symmetric. In the example below, using plus 3 standard deviations produces a risk level in percentiles of 98.6% as opposed to the desired risk level of 99.9%.



For example, for a Gamma Distribution (which can be used to model aggregate insurance claims) with the PDF of  $f(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$ , and  $\alpha = 2, \beta = 0.5$  (For this distribution, Mean =  $\frac{\alpha}{\beta} = \frac{2}{0.5} = 4$  and Standard Deviation =  $\sqrt{\frac{\alpha}{\beta^2}} = \sqrt{\frac{2}{0.25}} \approx 2.83$ ). The percentile values and the values from standard deviations of this Gamma Distribution are listed below:

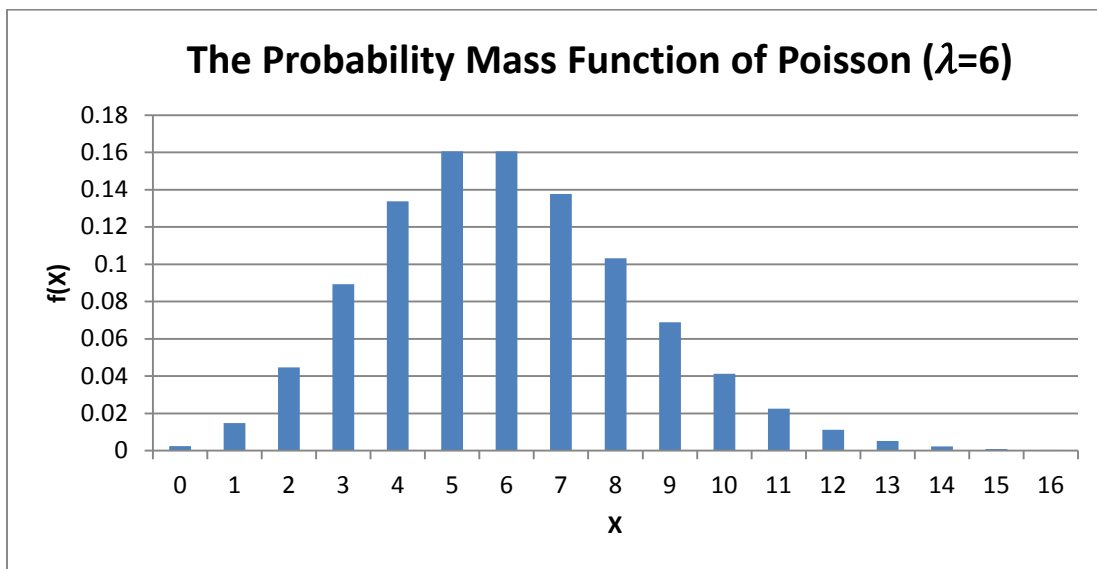
	Number of Standard Deviation	Value from Gamma	Risk Level from SD	Anticipated Risk Level
Scenario 1	3 Standard Deviations	12.49	*98.6%	**99.9%
Scenario 2	1 Standard Deviation	6.83	85.5%	84%
Scenario 3	Mean	4.00	59.4%	50%
Scenario 4	-1 Standard Deviation	1.17	11.7%	16%
Scenario 5	-3 Standard Deviations	-4.49	Not Defined	0.1%

\* 98.6% =  $F_{Gamma}(3 \text{ Standard Deviation})$ , Where  $Gamma(\alpha = 2, \beta = 0.5)$

\*\* 99.9% =  $\Phi(3 \text{ Standard Deviation})$

## 2. Adding or subtracting a number of Standard Deviations from the Mean can give invalid results

Some distributions, such as Gamma and Poisson Distribution, are defined only for non-negative numbers. However, if the standard deviation method were applied, there might be negative value, which is not defined in those distributions.





For example, for a Poisson Distribution (which can be used to model the number of deaths) with the  $\lambda = 6$  (Mean =  $\lambda = 6$  and Standard Deviation =  $\sqrt{\lambda} \approx 2.45$ ).

With a standard deviation approach, the Poisson value at the mean less three standard deviations could be negative. An example for this is a negative value for the mean less 3 standard deviations. For a Poisson Distribution,  $\lambda = \text{mean} = \text{variance}$ . If  $\lambda = 6$ , mean =6 and the standard deviation=  $\sqrt{\lambda} \approx 2.45$ .

The formula is Mean minus 3 times the Standard deviation.

$$6 - 3 \times \sqrt{\lambda} = 6 - 3 \times 2.45 \approx -1$$

This is an undefined value for the Poisson Distribution.

For those interested, the percentile values and the values from standard deviations of this Poisson Distribution are listed below:

	<b>Number of Standard Deviation</b>	<b>Value from Poisson</b>	<b>Risk Level from SD</b>	<b>Anticipated Risk Level</b>
Scenario 1	3 Standard Deviations	*13	99.6%	99.9%
Scenario 2	1 Standard Deviation	8	84.7%	84%
Scenario 3	Mean	6	60.6%	50%
Scenario 4	-1 Standard Deviation	4	28.5%	16%
Scenario 5	-3 Standard Deviations	-1	Not Defined (Poisson random variable must be Non-negative)	0.1%

$$13 \approx 6 + 3 \times 2.45 = 13.35 \quad (\text{Rounded to the nearest integer})$$



## Appendix B: Percentile Values of the Binomial Distribution

The Percentile Values of the Binomial Distribution, used in step 3 of the Binomial Distribution Example of the Percentile Methodology, are developed below.

Due to the Central Limit Theorem (CLT), the percentile value of the random variable  $X$  can be calculated as:

$$\text{Percentile Value} = E(X) + Z_{\alpha} \times \sqrt{\text{Var}(X)}$$

Where  $Z_{\alpha}$  is the Z score of the standard Normal Distribution.

$$E(X) = np = L \quad (L \text{ is the expected number of lapses})$$

$$\text{Var}(X) = np(1 - p) = L(1 - \text{LapseRate})$$

By substitution of terms:

$$\text{Percentile Value} = L + Z_{\alpha} \times \sqrt{L(1 - \text{LapseRate})}$$

## Appendix C: Percentile Values of the Poisson Distribution

The Upper and Lower Bounds of the Poisson Distribution, used in step 3 of the Poisson Distribution Example of the Percentile Methodology, are developed below.

Byar suggested an extremely accurate approximation to the exact Poisson test, which is obtained by calculating the deviate

$$\chi = \sqrt{9\tilde{A}} \left[ 1 - \frac{1}{9\tilde{A}} - \left( \frac{E}{\tilde{A}} \right)^{\frac{1}{3}} \right]$$

$\tilde{A} = A$  if  $A$  exceeds  $E$ .

$\tilde{A} = A + 1$  if  $A$  is less than  $E$ .

Where  $A$  is the observed number of deaths, and  $E$  is the expected number of deaths.

In addition, by applying the Wilson and Hilferty (1931) approximation for chi-square percentiles,

$$\chi_{v,\alpha}^2 = v \left( 1 - \frac{2}{9v} + Z_\alpha \sqrt{\frac{2}{9v}} \right)^3$$

Where  $v$  is the degree of freedom and  $\alpha$  is the risk level for the percentile value.  $Z_\alpha$  denotes the 100 $\alpha$  percentile of the standard Normal Distribution.

Thus, for a given risk level over 50% percentile value, the formula (Rothman & Boice, 1979) can be applied,<sup>4</sup>

$$Upper\ Bound = (A + 1) \times \left( 1 - \frac{1}{9(A + 1)} + \frac{Z_\alpha}{3\sqrt{A + 1}} \right)^3$$

For a given risk level less than 50% percentile values, the formula can be applied,

$$Lower\ Bound = (A) \times \left( 1 - \frac{1}{9A} + \frac{Z_\alpha}{3\sqrt{A}} \right)^3$$

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<sup>4</sup>N.E. Breslow & N.E. Day, *Statistical Methods in Cancer Research, Volume II – The Design and Analysis of Cohort Studies* (Publication No. 82, Lyon, International Agency for Research on Cancer, 1987), page 69.

[http://www.iarc.fr/en/publications/pdfs-online/stat/sp82/SP82\\_vol2-2.pdf](http://www.iarc.fr/en/publications/pdfs-online/stat/sp82/SP82_vol2-2.pdf).

## Appendix D: Academic Review

**Reviewer – Jay Vadiveloo, Ph.D, FSA, MAAA, CFA**

As an academician with many years of industry experience, and as Director of the Goldenson Center for Actuarial Research at the University of Connecticut, I focus on undertaking applied actuarial research projects which serve the needs of industry. I am always interested in practical approaches which have an academic basis to solve complex actuarial issues. This is one such paper and the authors have identified a practical alternative that can easily be used by companies to approximate an otherwise complex stochastic approach to calculating annuity reserves based on a principle based reserving methodology. The authors have developed this approach using fundamental statistical principles. Key risk drivers which impact the risks of a given product group are first identified. A set of five scenarios at different pre-set risk levels is determined. A statistical methodology is then used to translate these five scenarios into percentiles and their associated percentile values. Different probability distributions are used to model the various key risk drivers – the Normal distribution for renewal expenses, the Binomial distribution for lapses and the Poisson distribution for mortality. Based on these probability distributions, the percentiles and percentile values corresponding to the five scenarios are then calculated. Finally, reserves for each of the five risk levels are determined for each of the key risk drivers. A weighted average of these five risk level reserves is then calculated to establish an overall risk reserve for the key risk driver. The weights are based on the percentile levels for the five scenarios.

What is notable about this technique is the following:

- It is simple to implement in contrast to a full-blown stochastic approach
- It is founded on fundamental statistical principles
- It is a consistent application of the principle based reserving approach
- It naturally adapts to changes in a company's experience since the parameters of the underlying probability distributions for each key risk driver are based on a company's own experience.

In conclusion, as an academician and actuary who focuses on practical and applicable actuarial research, I applaud the authors for coming up with a useful, implementable, consistent and rigorous methodology to assist companies in complying with a principle based reserving approach for setting reserve levels. I believe this percentile methodology applied to the Representative Scenario Method could become the standard by which company-determined principle based reserves could be measured against by regulators and auditors.